Math 429 - Exercise Sheet 12

1. Check the Serre relation

$$\mathrm{ad}_{E_i}^{1-c_{ij}}(E_j) = 0$$

for the classical Lie algebras $\mathfrak{sl}_n, \mathfrak{o}_n, \mathfrak{sp}_{2n}$ (where E_i denote generators of the simple root spaces).

2. Check the well-definedness of the action in Proposition 30.

3. Prove the formula

$$[F_k, \mathrm{ad}_{E_i}^{1-c_{ij}}(E_j)] = 0 \tag{1}$$

in the Lie algebra $\widetilde{\mathfrak{g}}_C$, for any $i \neq j$ and k. In other words, show that the Lie brackets above are 0 by using only antisymmetry, the Jacobi identity, and relations (154)-(157). *Hint: show that*

$$ad_{E_i}^{1-c_{ij}}(E_j) = \sum_{t=0}^{1-c_{ij}} (-1)^t \binom{1-c_{ij}}{t} E_i^t E_j E_i^{1-c_{ij}-t}$$

in $U(\widetilde{\mathfrak{g}}_C)$*, and use this to prove* (1) *in* $U(\widetilde{\mathfrak{g}}_C)$*.*

4. In the lecture, we showed how to associate to any Dynkin diagram X a complex semisimple Lie algebra $\mathfrak{g}_X$. Show that if a Dynkin diagram Y contains another Dynkin diagram X inside it (by which we mean that X contains as many edges between any two of its vertices as Y did), then there is an injective homomorphism $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$ between the corresponding Lie algebras.

(*) With the notation from the lecture notes, prove that $\mathfrak{i}$ is contained in any ideal of $\widetilde{\mathfrak{g}}_C$ that has finite codimension (i.e. $\mathfrak{g}_C$ is the largest finite-dimensional quotient of $\widetilde{\mathfrak{g}}_C$).